Fractional spaces and conservation laws

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Abstract In 1994, Lions, Perthame and Tadmor conjectured the maximal smoothing effect for multidimensional scalar conservation laws in Sobolev spaces. For strictly smooth convex flux and the one-dimensional case we detail the proof of this conjecture in the framework of Sobolev fractional spaces $W^{s,1}$, and in fractional BV spaces: BV^s. The BV^s smoothing effect is more precise and optimal. It implies the optimal Sobolev smoothing effect in $W^{s,1}$ and also in $W^{s,p}$ with the optimal $p = 1/s$. Moreover, the proof expounded does not use the Lax-Oleinik formula but a generalized one-sided Oleinik condition.

1 Introduction

In this short note we first prove a remark on the optimal smoothing effect in Sobolev spaces given by Lions, Perthame and Tadmor [LPT] for strictly convex flux with a power-law type of degeneracy. Then we improve this result in fractional BV spaces: BV^s. The smoothing effect in BV^s with the optimal s yields to the optimal Sobolev regularity in $W^{s,p}$ with the same exponent s and the optimal p . This BV^sregularity also gives traces property of the entropy solution ([BM, DOW, DR, P1, P2]). We do not use the Lax-Oleinik formula ([E]) as suggested in [LPT] and used in [BGJ, CJ3] but the one-sided Lipschitz condition on the velocity of the entropy solution (2) below. In all the paper, the following one-dimensional scalar conservation law are considered:

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$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = 0 \ \text{ on } \mathbb{R}\times]0,+\infty[\\ \\ u(x,0) = u_0(x) \ \text{ on } \mathbb{R} \end{cases}, \tag{1}$$

where the flux f is strictly convex and $f \in C^1$, that is to say that the velocity $a(u) = f'(u)$ is strictly increasing. Lax ([La]) and Oleinik ([O]) obtained the optimal smoothing effect in BV for an uniformly convex flux: $\inf f'' > 0$. The BV regularity is known to be lost if the second derivative of the flux vanishes at only one point ([Ch1]). Nevertheless we can obtain BV^s regularity with $0 < s < 1$. BV^s spaces are essentially fractional BV spaces with similar properties as for example $BV = BV^1$. First applications of such fractional BV spaces for a 2×2 system are already obtained in [BGJP].

We do not speak about SBV regularity ([AGV2, AD, B]) since this related to an uniform convex flux and the BV smoothing effect. On the other hand there are many other studies of the regularizing effect ([AGV1, BM, Cc, Ja]) which can be interpreted in the framework of fractional BV spaces but are beyond the scope of this short note.

The paper is organized as follow. In Section 2 a one-sided Lipschitz condition is recalled and linked with the degeneracy of the flux. Then, the smoothing effect is stated in $W^{s,p}$ and BV^s. Section 3 gives a very simple proof of the smoothing effect in BV^s and some consequences as the optimal $W^{s,p}$ regularity. We also present in the final Section 4 an alternative proof of the optimal Sobolev regularity in $W^{s,1}$.

2 One-sided condition and degeneracy of the flux

The proof uses the following generalized one-sided Oleinik inequality due to Dafermos ([D]) (see also [Ch2]) under convex condition on the flux and for almost all $x < y$:

$$a(u(y,t)) - a(u(x,t)) \leq \frac{y-x}{t} \tag{2}$$

There is a geometric interpretation of Inequality (2). If there is no shock issued from $(x,0)$ and $(y,0)$ in the (x,t) plane then u is constant on the two characteristics issued from $(x,0)$ and $(y,0)$ and the constant difference of the velocities is less than the constant velocity needed to meet the characteristics at time t. Inequality (2) is usually interpreted as an entropy condition: the jump of velocity can only be negative when $y \to x$. A complete argument can be given with the generalized characteristics of Dafermos ([D]).

Since any strictly C^1 convex flux can be approached in C^1 by a sequence of C^2 uniformly convex flux, inequality (2) is still valid for entropy solutions almost everywhere. If the flux is uniformly convex then $m|u-v| \leq |a(u) - a(v)|$ for a positive constant m and Inequality (2) yields to the one-sided Oleinik condition:

$u(y,t)-u(x,t) \leq \frac{y-x}{mt}$ which can be rewritten with $(v)_+ = \max(v,0)$ for almost all $(x,y) \in \mathbb{R}^2$ and $t>0$:

$$(u(y,t)-u(x,t))_+ \leq \frac{(y-x)_+}{mt}$$

For a non uniformly convex flux, the previous one-sided Lipschitz condition becomes a one-sided Hölder condition: Inequality (5) at the end of this section. Before, a precise definition of a nonlinear flux is needed.

A power-law type of nonlinear degeneracy is considered as in [BGJ]. The following condition is enough for a strictly convex flux with some regularity and power-law flux for instances.

Definition 1. Let $f \in C^1(K,\mathbb{R})$, where K is a closed interval of $\mathbb{R}$. We say that the degeneracy of f on K is at least $q>0$ if the continuous derivative $a(u) = f'(u)$ satisfies :

$$\inf_{(u,v)\in(K\times K)\smallsetminus\mathscr{D}_K} \frac{|a(u)-a(v)|}{|u-v|^q} > 0, \tag{3}$$

where $\mathscr{D}_K = \{(u,v) \in (K\times K) \mid u \neq v\}$. The lowest real number q, if there exists, is called the degeneracy of f on K and denoted p.

Inequality 3 is equivalent to:

$$\exists m>0, \forall (u,v) \in K^2, |a(u)-a(v)| \geq m\,|u-v|^q. \tag{4}$$

In particular, (4) implies that $a: K \to \mathbb{R}$ is strictly monotonic, since a is injective and continuous.

Example 1. If $f(u) = |u|^{1+p}$, $p \geq 1$, then p is the degeneracy of f on any interval which contains 0.

Remark 1. 1) Suppose that for all $u,v \in K$, $|a(u)-a(v)| \geq m\,|u-v|^q$ and that a is differentiable at u_0. Then $q \geq 1$, since $\infty > \left|\frac{a(u)-a(u_0)}{u-u_0}\right| \geq m\,|u-u_0|^{q-1}$.

2) If f is smooth on K, then f has a degeneracy p which is a positive integer. ([BGJ])

Replacing $f(u)$ by $-f(-u)$ if necessary, we will assume subsequently that $a(u) = f'(u)$ is strictly increasing, so f is strictly convex.

Now, one-sided condition (2) on the velocity is interpreted as the following one-sided Hölder condition for almost all x,y:

$$(u(y,t)-u(x,t))_+ \leq K\frac{[(y-x)_+]^s}{t^s}, \tag{5}$$

where $s = \frac{1}{p}$, $K = m^{-s}$, $(v)_+ = \max(v,0)$ since $m(u-v)^p = m((u-v)_+)^p \leq a(u)-a(v)$ for $u \geq v$. Again, the exponent s naturally appears to estimate the positive variation of the entropy solution u .

Now, we can state the smoothing effect for entropy solutions.

Theorem 1. *Let u_0 belongs to $L^\infty(\mathbb{R})$ function, K be the convex hull of $u_0(\mathbb{R})$, f a C^1 strictly convex flux with degeneracy p on K. Then the associated entropy solution of the conservation law (1) have got the following regularity in space for all positive time t and for all $\varepsilon > 0$: $u(.,t) \in W^{s-\varepsilon,1/s}_{loc}(\mathbb{R}) \cap BV^s_{loc}(\mathbb{R})$.*

The regularity in $W^{s-\varepsilon,1}_{loc}(\mathbb{R})$ was conjectured in [LPT] and proved in [Ja]. The regularity in BV^s was first proved in [BGJ] . The regularity in $W^{s-\varepsilon,1/s}_{loc}$ is the consequence of the BV^s_{loc} regularity. The optimality in Sobolev spaces can be found in [DW] and in the fractional BV spaces in [CJ1].

The main originality of this note is hence to give new and simpler proofs for this smoothing effect. Our proofs are based on the key BV estimate of the velocity $a(u)$ for positive time given by (2). This is obviously a nonlinear regularity estimate on the entropy solution u which can easily be translated into more traditional regularity estimates.

The spaces BV^s are in particular well adapted for this, leading to a very simple proof in the next section. From this BV^s regularity it is straightforward to deduce fractional Sobolev regularity as well. But of course one can also prove directly the Sobolev regularity; we give an example of such a proof in the last Section.

3 BV^s smoothing effect

We can define BV^s_+ as $W^{s,1}_+$ and try to adapt the previous proof from Section 4.3. Unfortunately, the equality: $L^\infty \cap BV^s_+ = BV^s$ for $0 < s < 1$ is an open problem. It is only known for $s = 1$ and, fortunately, it is enough to get the optimal BV^s regularity.

3.1 BV^s spaces

We recall briefly the definition and the main properties of fractional BV spaces.

Definition 2. Let I be an non-empty interval of $\mathbb{R}$ and let $\mathscr{S}(I)$ be the set of subdivisions of I: $\mathscr{S}(I) = \{(x_0, x_1, ..., x_n), n \geq 1, x_i \in I, x_0 < x_1 < ... < x_n\}$.

For $0 < s \leq 1$ set

$$TV^s u[I] = \sup_{\mathscr{S}(I)} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})|^{\frac{1}{s}},$$

then $BV^s(I) := \{u, TV^s u[I] < \infty\}$.

For $s = 1$ we recover the space BV of functions of bounded variation. Functions in BV^s have always left and right traces, like in BV ([MO]). The exponent s is related to the fractional Sobolev derivative: $BV^s \subset W^{s-\varepsilon,1/s}$ for all $\varepsilon > 0$ ([BGJ]). The space BV^s is also called the space of functions of bounded p -variation, with $p = \frac{1}{s}$.

3.2 The short proof of the BV^s smoothing

Set $TV_+u := \sup_{\substack{n\in\mathbb{N}^* \\ x_0<x_1<\ldots<x_n}} \sum_{i=1}^{n}(u(x_i)-u(x_{i-1}))_+$ and $BV_+ := \{u, TV_+u<\infty\}$
then $a(u(.,t)) \in BV_+$ from inequality (2). According to the Maximum principle $a(u(x,t)) \in L^\infty$ as the initial data u_0. Since $BV_+ \cap L^\infty = BV$, then $a(u(.,t)) \in BV$. Moreover, the velocity $a()$ has at most a power law degeneracy:

$$|a(u)-a(v)| \geq m\,|u-v|^p\,,$$

thus $u \in BV^s$, where $s = \frac{1}{p}$.

This proof is very short and shortens the proof given in [BGJ]. Moreover, it gives more information about the singularity of u. For instance if the flux is a convex power law $f(u) = |u|^{1+p}$ and the convex hull K of $u_0(\mathbb{R})$does not contain the singular point 0 then the entropy solution associated to the initial date u_0 belongs to BV. That means that the BV^s regularity is due to bigger oscillations around the state $u=0$. Notice that since there is only a finite number of oscillations with any given positive strength thus the oscillations near state $u=0$ has to be smaller and smaller and with infinitely many oscillations as optimal examples given in [CJ1, DW].

4 Optimal smoothing effect in Sobolev spaces $W^{s,1}$

The best smoothing effect in Sobolev spaces $W^{s,1}$ was suggested in [LPT] with the Lax-Oleinik formula, bounded in [DW] and proved in [Ja] with a kinetic formulation and a BV assumption on the velocity. In this short note an another proof is proposed.

We recall a classical result for $W_+^{1,1}$ and BV_+. Then a similar result in $W_+^{s,1}$ is proved and used to get the maximal smoothing effect for conservation laws.

4.1 Usual results in $W^{1,1}$

$W_+^{1,1}(\mathbb{R})$ is the set of functions u such that the semi-norm $|u|_1^+ = \int_{\mathbb{R}} (\partial_x u)_+\,dx$ is finite, where $(v)_+ = \max(v,0)$.

Lemma 1. $W_+^{1,1}(\mathbb{R}) \cap L^1(\mathbb{R}) \subset BV(\mathbb{R})$.

Notice that $W_+^{1,1}(\mathbb{R}) \cap L^1(\mathbb{R})$ is bigger than $W^{1,1}(\mathbb{R})$: for instance consider the BV function $u(x) = x\,\chi_{[0,1](x)}$ where χ_I is the indicator function of the set I.

The lemma follows from the equality $\int_{\mathbb{R}} |\partial_x u|\,dx = 2\,|u|_1^+$, which is valid for any smooth compactly supported function, since $|x| = 2x_+ - x$ and then $\int_{\mathbb{R}} |\partial_x u|\,dx = 2\int_{\mathbb{R}} (\partial_x u)_+\,dx - \int_{\mathbb{R}} \partial_x u\,dx = 2\int_{\mathbb{R}} (\partial_x u)_+\,dx$.

The space BV is better fitted through the control of the positive variation since

$$BV_+(\mathbb{R}) \cap L^\infty(\mathbb{R}) = BV(\mathbb{R})$$

where $BV_+(\mathbb{R})$ is the space of function u such that

$$TV_+ u = \sup_{n\in\mathbb{N}, x_0<x_1<\cdots<x_n} \sum_{i=1}^{n} (u(x_i) - u(x_{i-1}))_+ < \infty.$$

4.2 The $W_+^{s,1}(\mathbb{R})$ case

The usual Sobolev semi-norm in $W^{s,1}(\mathbb{R})$ for $s \in]0,1[$ is :

$$|u|_s = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x)-u(y)|}{|x-y|^{1+s}} dx dy = \int \int \frac{|u(x+h)-u(x)|}{|h|^{1+s}} dx dh.$$

The Sobolev semi-norm in $W_+^{s,1}$ for $s \in]0,1[$ is :

$$|u|_s^+ = \int_{\mathbb{R}} \int_{h>0} \frac{[u(x+h)-u(x)]_+}{|h|^{1+s}} dx dh.$$

The definition of the set $W_+^{s,1}(\mathbb{R})$ is:

Definition 3. $u \in W_+^{s,1}(\mathbb{R})$ if $\displaystyle\iint_{x>y} \frac{[u(x)-u(y)]_+}{|x-y|^{1+s}} dx dy < +\infty$.

Proposition 1. *For all* $0 < s < 1$, $L^1(\mathbb{R}) \cap W_+^{s,1}(\mathbb{R}) = W^{s,1}(\mathbb{R})$.

For $s < 1$ the estimates for the "one-sided" semi-norm is almost enough to stay in the associated Sobolev space. This is not true for $s = 1$ (Lemma 1). Proposition 1 is also valid locally: $L^1_{loc}(\mathbb{R}) \cap W^{s,1}_{+,loc}(\mathbb{R}) = W^{s,1}_{loc}(\mathbb{R})$.

Proof. The bound of the Sobolev semi-norm $2I = \iint \frac{|u(x)-u(y)|}{|x-y|^{1+s}} dx dy < +\infty$ is obtained thanks to the integral $I = \iint_{x>y} \frac{|u(x)-u(y)|}{|x-y|^{1+s}} dx dy$, which is a priori singular. Consider for $\varepsilon > 0$ the well-defined integrals :

$$\begin{aligned} I_\varepsilon^+ &= \iint_{x>y} \frac{[u(x)-u(y)]_+}{\varepsilon + |x-y|^{1+s}} dx dy \\ I_\varepsilon^- &= \iint_{x>y} \frac{[u(x)-u(y)]_-}{\varepsilon + |x-y|^{1+s}} dx dy \end{aligned},$$

$$
\begin{aligned}
I_\varepsilon &= \iint_{x>y} \frac{|u(x)-u(y)|}{\varepsilon+|x-y|^{1+s}}\,dx\,dy = I_\varepsilon^+ + I_\varepsilon^- \\
J_\varepsilon &= \iint_{x>y} \frac{u(x)-u(y)}{\varepsilon+|x-y|^{1+s}}\,dx\,dy = I_\varepsilon^+ - I_\varepsilon^-
\end{aligned}.
$$

Since

$$
\begin{aligned}
J_\varepsilon &= \int u(x) \int_{x>y} \frac{1}{\varepsilon+|x-y|^{1+s}}\,dy\,dx - \int u(y) \int_{x>y} \frac{1}{\varepsilon+|x-y|^{1+s}}\,dx\,dy \\
&= \int u(x) \int_{x>y} \frac{1}{\varepsilon+|x-y|^{1+s}}\,dy\,dx - \int u(x) \int_{x<y} \frac{1}{\varepsilon+|x-y|^{1+s}}\,dy\,dx \\
&= \int u(x) \int_{h>0} \frac{1}{\varepsilon+|h|^{1+s}}\,dh\,dx - \int u(x) \int_{h<0} \frac{1}{\varepsilon+|h|^{1+s}}\,dh\,dx \\
&= 0
\end{aligned},
$$

it follows that $I_\varepsilon^+ = I_\varepsilon^-$ and then $I_\varepsilon = 2I_\varepsilon^+ \leq 2I_0^+ < +\infty$. The monotone convergence theorem yields $\lim_{\varepsilon\to 0} I_\varepsilon = I = 2I_0^+$, so that $u \in W^{s,1}(\mathbb{R})$.

4.3 Optimal smoothing effect in fractional Sobolev spaces

Thanks to Proposition (1), it suffices to get a $W_+^{\sigma,1}$ estimate to have the $W^{\sigma,1}$regularity $\forall\,\sigma < s$. The information is propagated with a finite speed for conservation laws so we work only locally with local Sobolev semi-norm as in [Ju].

The well known embedding from Holder spaces to Sobolev spaces: $W_{loc}^{s,\infty} \subset W_{loc}^{\sigma,1}, \forall \sigma < s$, is simply extended to $W_{+,loc}^{s,\infty}$:

Lemma 2. $W_{+,loc}^{s,\infty} \subset W_{+,loc}^{\sigma,1}, \forall \sigma < s,$

Proof. Assume that $(u(x+h)-u(x))_+ \leq Ch^s$ for $h > 0$.

Let h_0 and A be some positive constants to estimate the $W_{+,loc}^{\sigma,1}$ semi-norm on the interval $]-A, A+h_0[$:

$$
|u|_{\sigma,loc}^+ = \int_{|x|<A} \int_{h_0>h>0} \frac{[u(x+h)-u(x)]_+}{h^{1+\sigma}}\,dx\,dh \leq \int_{|x|<A} \int_{h_0>h>0} C\,h^{s-\sigma-1}\,dx\,dh
$$

$$
|u|_{\sigma,loc}^+ \leq 2AC \int_{h_0>h>0} h^{s-\sigma-1}\,dh
$$

which is finite if and only if $\sigma < s$.

Finally, the one-sided Holder condition for an entropy solution is already given by Inequality (5) and the Sobolev regularity follows since an entropy solution is bounded in L^∞ as its initial data and $W_{+,loc}^{\sigma,1} \cap L^\infty \subset W_{loc}^{\sigma,1}$.

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